TURBULENT BOUNDARY LAYER ON A PLATE IN AN INCOMPRESSIBLE LIQUID WITH MASS INJECTION

I. P. Ginzburg and N. S. Krest'yaninova

Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 4, pp. 444–450, 1965

UDC 532.517.4

Within the limits of a semi-empirical theory based on a two-layer system, as proposed in [5], an examination is made of the influence of mass addition, homogeneous with the main flow, on friction and the parameters of the turbulent boundary layer (thickness of the laminar sublayer, velocity at the edge of the laminar sublayer, etc.).

The influence of mass addition on surface friction and heat transfer in the turbulent-boundary-layer case is examined in [1–5] and elsewhere. In the solution of the corresponding equations, additional assumptions of one kind or another were made regarding the thickness of the laminar sublayer or the velocity at its edge. The present paper investigates the influence of mass addition on the parameters of the boundary layer and on friction, on the basis of the general relations of a two-layer system in the semiempirical theory of turbulence.

To evaluate the validity of the limiting laws developed in [6], and also for simplicity of calculation, the case of an incompressible liquid is examined here. More complicated problems will be examined in later papers.

To obtain an approximate solution of the problem posed, we shall assume that the friction stress may be represented as a polynomial in the dimensionless velocity $\varphi = v_x/u$, this being

$$\tau = \tau_w (1 + A_1\varphi + A_2\varphi^2 + A_3\varphi^3). \tag{1}$$

We shall determine coefficients A_1, A_2, and A_3 from the boundary conditions, using the equations of motion

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0; \quad v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y}. \tag{2}$$

From (2), we find that, when $\varphi = 0$

$$\frac{\partial \tau}{\partial \varphi} = \rho v_n u, \quad \frac{\partial^2 \tau}{\partial \varphi^2} = 0. \tag{3}$$

Using (3), and also the condition that, when $\varphi = 1$, $\tau = 0$, we obtain (1) in the form

$$\tau = \tau_w \left[1 + \frac{\rho v_w u}{\tau_w}\varphi - \left(1 + \frac{\rho v_w u}{\tau_w}\right)\varphi^3\right]. \tag{4}$$

In accordance with the basic relations of the semi-empirical theory of turbulence, we shall assume that

$$\tau = k^2 \rho y^2 u^2 \left(\frac{\partial \varphi}{\partial y}\right)^2 \tag{5}$$

when $y \geq \delta_L$ or $\varphi \geq \varphi_L$, and when $y \leq \delta_L$

$$\tau = \mu u \, \partial\varphi/\partial y. \tag{6}$$

The thickness of the laminar sublayer is determined from the condition

$$\left.\frac{\partial \varphi}{\partial y}\right|_{y=\delta_L - 0} = k_1 \left.\frac{\partial \varphi}{\partial y}\right|_{y=\delta_L + 0}. \tag{7}$$

From (7), taking (5) and (6) into account, we obtain

$$\delta_L = \frac{k_1}{k}\frac{\nu}{\sqrt{\tau_L/\rho}}. \tag{8}$$

Substituting τ from (4) into (8), we have

$$\delta_L = \frac{k_1}{k}\nu\left\{\frac{\tau_w}{\rho}\left[1 + \frac{\rho v_w u}{\tau_w}\varphi - \left(1 + \frac{\rho v_w u}{\tau_w}\right)\varphi^3\right]\right\}^{-1/2}. \tag{9}$$

We shall use (4) and (6) to determine φ_L. From them we have

$$\tau_w\left[1 + \frac{\rho v_w u}{\tau_w}\varphi - \left(1 + \frac{\rho v_w u}{\tau_w}\right)\varphi^3\right] = \mu u \frac{\partial \varphi}{\partial y}.$$

Integrating the last equation and neglecting φ^3, we obtain

$$\frac{\tau_w}{\mu u} y = A^{-1}\ln(1 + A\varphi) \quad \text{for} \quad y \leqslant \delta_L, \tag{10}$$

where $A = \rho v_w u/\tau_w$.

Assuming $y = \delta_L$, $\varphi = \varphi_L$ from (10), we obtain a second equation for determing δ_L and φ_L:

$$\tau_w \delta_L/\mu u = A^{-1}\ln(1 + A\varphi_L). \tag{11}$$

We shall introduce the relative values $\delta_L^* = \delta_L/\delta_{L0}$, $\varphi_L^* = \varphi_L/\varphi_{L0}$, $c_f^* = c_f/c_{f0}$, where δ_{L0}, φ_{L0}, c_{f0} are corresponding values of δ_L, φ_L and of the local friction coefficient $c_f = 2\tau_w/\rho u^2$ when there is no blowing (A = 0). Then, neglecting φ^3, we may write (9) and (10) in the form

$$\delta_L^* = (c_f^* + N\varphi_{L0}\varphi_L^*)^{-1/2}, \tag{12}$$

$$\frac{N\varphi_{L0}}{\sqrt{c_f^*}} = \sqrt{1 + \frac{N\varphi_{L0}\varphi_L^*}{c_f^*}}\,\ln\left(1 + \frac{N\varphi_{L0}\varphi_L^*}{c_f^*}\right), \tag{13}$$

where

$$N = 2v_w/c_{f0}u, \quad \varphi_{L0} = (k_1/k)(c_{f0}/2)^{-1/2}.$$

The last equation enables us to evaluate φ_L^* as a function of N, φ_{L0} if c_f^* is known. To determine c_f^* we shall first obtain an expression for the velocity profile in the turbulent core by simultaneous solution of (4) and (5). We shall limit φ to the first degree in (4); integrating (5) with the conditions $y = \delta$, $\varphi = 1$, we obtain the velocity profile when $y \geq \delta_L$ in the form

$$\frac{N}{2k}\sqrt{\frac{c_{f0}}{2}}\ln\left(\frac{y}{\delta}\right) = \sqrt{c_f^* + N\varphi} - \sqrt{c_f^* + N}. \quad (14)$$

Putting $\varphi = \varphi_L$, $y = \delta_L$ in (14), we obtain

$$\frac{\delta}{\delta_L} = \exp\left[\frac{2k_1}{N\varphi_{L0}}\left(\sqrt{c_f^* + N} - \sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}\right)\right]. \quad (15)$$

In the absence of blowing,

$$\delta_0/\delta_{L0} = \exp\left[k_1(1-\varphi_{L0})/\varphi_{L0}\right]. \quad (16)$$

Taking (12) and (16) into account, (15) may be written as

$$\delta^* = \frac{1}{\sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}}\exp\left[\frac{2k_1}{N\varphi_{L0}}\left(\sqrt{c_f^* + N} - \sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}\right) + \frac{k_1(\varphi_{L0} - 1)}{\varphi_{L0}}\right]. \quad (17)$$

In order to close the system of equations (12), (13), and (17) which determine δ_L^*, φ_L^*, c_f^* and δ^*, we shall use the von Karman integral relation, which, in the case of a plate with blowing, may be written as follows:

$$d\Theta/dx = c_f/2 + b, \quad (18)$$

where

$$b = \frac{v_w}{u}, \quad \Theta = \int_0^\delta \frac{v_x}{u}\left(1 - \frac{v_x}{u}\right)dy.$$

We shall first determine Θ/δ. We have

$$\Theta/\delta = \int_0^1 \varphi(1-\varphi)\,d(y/\delta).$$

It follows from (14) that

$$\varphi = 1 + \frac{\varphi_{L0}}{k_1}\sqrt{c_f^* + N}\,\ln\left(\frac{y}{\delta}\right) + \frac{b}{4k^2}\ln^2\left(\frac{y}{\delta}\right).$$

Substituting this expression for the velocity in the preceding relation and performing the integration, we obtain

$$\frac{\Theta}{\delta} = \left[\left(1 + 3\frac{N\varphi_{L0}^2}{k_1^2}\right)\left(\frac{\varphi_{L0}}{k_1}\sqrt{c_f^* + N} - \frac{N\varphi_{L0}^2}{2k_1^2}\right) - 2\frac{\varphi_{L0}^2}{k_1^2}\left(c_f^* + N\right)\right]. \quad (19)$$

When $N \to 0$ we obtain

$$\frac{\Theta_0}{\delta_0} = \frac{\varphi_{L0}}{k_1}\left(1 - 2\frac{\varphi_{L0}}{k_1}\right). \quad (20)$$

Dividing (19) by (20) and using (7), we shall find $\Theta^* = \Theta/\Theta_0$. Introducing Θ^* into (18) and going over to a new independent variable φ_{L0}, we obtain

$$\frac{d\Theta^*}{d\varphi_{L0}} = -\left(\frac{k_1}{\varphi_{L0}^2} + \frac{2}{k_1 - 2\varphi_{L0}}\right)\left(c_f^* + N - \Theta^*\right). \quad (21)$$

If we replace Θ^* in the last equation by its value obtained from (19), (20), and (17), eliminating φ_L^* with the aid of (13), we obtain an ordinary nonlinear differential equation for determining c_f^* as a function of φ_{L0} and of the blowing parameter N. Here N may be both a constant and a function of φ_{L0}.

The dependence of φ_{L0} on Re_x is found from the known relations which are easily obtained from (16),

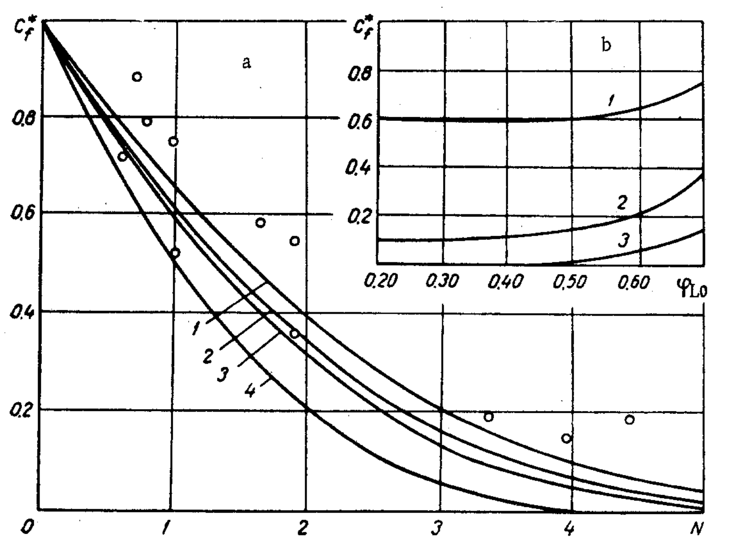

Fig. 1. Dependence of the friction coefficient on the blowing parameter: a) with $\varphi_{L0} = 0.60$ and $Re_x = 9.55 \cdot 10^4$ (1); 0.45 and 2.24×10^6 (2); 0.35 and 6.32×10^7 (3); $\varphi_{L0} \to 0$ and $Re_x \to \infty$ (4); b) with N = 1 (1), 2 (2), 3 (3).

(18), and (20). Equation (21) has been solved numerically, taking account of (13), (17), (19), and (20), for two cases: When N = const and when b = const.

Because of the undetermined initial conditions at the front of the plate, blowing was predicated with $\varphi_{LI} = 0.70$ (φ_{LI} corresponds to φ_{L0} at which blowing begins). The results of the solution for N = const are shown in Figs. 1–3.

Figure 1a shows the dependence of relative friction coefficient on the blowing parameter. It comprises a family of curves with parameter φ_{L0} (or Re_x). Also shown is the curve corresponding to the limiting laws [6], and the experimental points of Mickley [7], as corresponding most closely to the conditions of the calculation ($M_\infty \to 0$; $T_w/T \to 1$; $Re_x \sim$ $\sim 10^4 - 10^6$). It may be seen from Fig. 1a that the results of the calculation according to the method described for finite Re_x are closer to the experimental points than the result of [6], while both coincide in the limiting case of $Re_x \to \infty$. From Fig. 1b, which shows the dependence of c_f^* on φ_{L0}, it may be seen that, at points near the origin of blowing, c_f^* depends very much on φ_{L0}, while as the distance from the origin of blowing increases, the dependence of c_f^* on φ_{L0} becomes weaker, and when $\varphi_{L0} < 0.45$, it may be assumed that the relative local friction coefficient does not depend on φ_{L0}. This information may be used in formulating an approximate solution of (21). In the situation when c_f^* = const and N = const along the plate, the solution of (21) may be written as

$$\Theta^* = c_f^* + N\left[1 - \frac{k_1 - 2\varphi_{LH}}{k_1 - 2\varphi_{L0}} \exp\left(k_1 \frac{\varphi_{L0} - \varphi_{LH}}{\varphi_{L0}\varphi_{LH}}\right)\right]. \quad (22)$$

Then c_f^* may be found from

$$c_f^* + N\left[1 - \frac{k_1 - 2\varphi_{LH}}{k_1 - 2\varphi_{L0}} \exp\left(k_1 \frac{\varphi_{L0} - \varphi_{LH}}{\varphi_{L0}\varphi_{LH}}\right)\right] =$$
$$= \left\{k_1^2\left[\left(1 + 3\frac{N\varphi_{L0}^2}{k_1^2}\right)\left(\frac{\varphi_{L0}}{k_1}\sqrt{c_f^* + N} - \frac{N\varphi_{L0}^2}{2k_1^2}\right) - 2\frac{\varphi_{L0}^2}{k_1^2}\left(c_f^* + N\right)\right]\right\} \times$$
$$\times \left\{\varphi_{L0}(k_1 - 2\varphi_{L0})\sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}\right\}^{-1} \times$$
$$\times \exp\left[\frac{2k_1}{N\varphi_{L0}}\left(\sqrt{c_f^* + N} - \sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}\right) - \frac{k_1(1 - \varphi_{L0})}{\varphi_{L0}}\right],$$

where φ_L^* is determined from (13). Graphical solution of (23) gives values of φ_L^* near the origin of blowing, differing from the exact solution values by less than 10%. As the distance from the origin of blowing increases, so too does the accuracy of agreement of the two solutions, and, for $\delta_L^* = 0.40$ and under, it reaches tenths of a percent. If the value of b remains constant along the plate, the solution of (22) may be written in the form

$$\Theta^* = c_f^* + \left(\frac{k_1}{k}\right)^2 \frac{b}{\varphi_{L0}^2}\left[1 - 2\frac{\varphi_{L0}}{k_1}\frac{k_1 - 3\varphi_{L0}}{k_1 - 2\varphi_{L0}}\right] - \left(\frac{k_1}{k}\right)^2 \frac{b(k_1 - 2\varphi_{LH})}{\varphi_{LH}^2(k_1 - 2\varphi_{L0})} \times$$
$$\times\left[1 - 2\frac{\varphi_{LH}(k_1 - 3\varphi_{LH})}{k_1(k_1 - 2\varphi_{LH})}\right]\exp\left[k_1\frac{\varphi_{L0} - \varphi_{LH}}{\varphi_{L0}\varphi_{LH}}\right]. \quad (24)$$

We may also determine c_f^* approximately from the equation

$$k_1\left[\left(1 + \frac{3b}{k^2}\right)\left(\frac{\varphi_{L0}}{k_1}\sqrt{c_f^* + \frac{bk_1^2}{k^2\varphi_{L0}^2}} - \frac{b}{2k^2}\right) - 2\frac{\varphi_{L0}^2}{k_1^2}\left(c_f^* + \frac{bk_1^2}{k^2\varphi_{L0}^2}\right)\right]\left[\varphi_{L0}(k_1 - 2\varphi_{L0})\sqrt{c_f^* + \frac{bk_1^2}{k^2\varphi_{L0}}\varphi_L^*}\right]^{-1} \times$$
$$\times \exp\left[\frac{2k^2\varphi_{L0}}{bk_1}\left(\sqrt{c_f^* + \frac{bk_1^2}{k^2\varphi_{L0}^2}} - \sqrt{c_f^* + \frac{bk_1^2}{k^2\varphi_{L0}}\varphi_L^*}\right) - \frac{k_1(1 - \varphi_{L0})}{\varphi_{L0}}\right] = c_f^* + \left(\frac{k_1}{k}\right)^2\frac{b}{\varphi_{L0}^2}\left[1 - 2\frac{\varphi_{L0}}{k_1}\frac{k_1 - 3\varphi_{L0}}{k_1 - 2\varphi_{L0}}\right] -$$
$$- \left(\frac{k_1}{k}\right)^2\frac{b(k_1 - 2\varphi_{LH})}{\varphi_{LH}^2(k_1 - 2\varphi_{L0})} \times \left[1 - 2\frac{\varphi_{LH}(k_1 - 3\varphi_{LH})}{k_1(k_1 - 2\varphi_{LH})}\right]\exp\left[k_1\frac{\varphi_{L0} - \varphi_{LH}}{\varphi_{L0}\varphi_{LH}}\right].$$

Variation of the Ratio $\ln\frac{\delta}{\delta_л} / \ln\frac{\delta_0}{\delta_{л0}}$ with Change of the Blowing Parameters

φ_{L0}	$\ln\frac{\delta}{\delta_L} / \ln\frac{\delta_0}{\delta_{L0}}$ for N		
	1.0	3.0	5.0
0.600	1.11	1.23	1.38
0.550	1.08	1.19	1.27
0.450	1.05	1.12	1.17
0.350	1.03	1.06	1.06

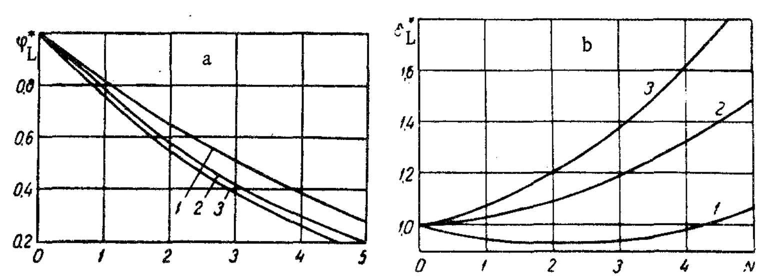

Fig. 2. Dependence of φ_L^* (a), and of δ_L (b) on N: 1–3) see Fig. 1a.

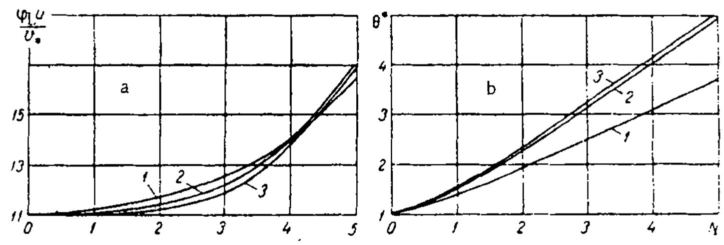

Fig. 3. Dependence of $\varphi_L / \frac{v_*}{u}$ (a) and Θ^*(b) on N: 1–3) see Fig. 1a.

The accuracy of agreement of the approximate and exact solutions may be said to be the same also when N = const along the plate.

It may be seen from Fig. 2 that φ_L^* always decreases with N, and does so faster, the larger R_x is; also, as far as δ_L^* is concerned, when $Re_x \sim 10^4$ and N is small, δ_L^* first decreases and then increases with increasing N. With $Re_x \sim 10^6$, for any N, an increase in the thickness of the laminar sublayer is observed. This one-sided variation of thickness of the laminar sublayer is evidently due to a twofold influence of blowing. On the one hand, it makes the laminar sublayer turbulent, and on the other hand, by decreasing the friction, it favors an increase of sublayer. It is suggested in some papers [3, 4] that $\varphi_L \sim v_*/u$ and that the ratio $\varphi_L / \frac{v_*}{u}$ remains constant during blowing and equal to its value in the absence of blowing.

It may be seen from Fig. 3a that the ratio $\varphi_L / \frac{v_*}{u}$ increases with N, particularly when N > 3. Fig. 3b gives Θ^* as a function of N and φ_{L0}. The figure shows that the momentum loss thickness increases with increase of the blowing parameter.

In order to evaluate possible limiting laws, values of $\ln\frac{\delta}{\delta_L} / \ln\frac{\delta_0}{\delta_{L0}}$ are presented in the table, from which it may be seen that this ratio is close to 1 and that, beginning from $\varphi_{L0} \leq 0.45$, it varies negligibly with N. This enables us to suggest the following simplified solution to the problem of the influence of blowing on the parameters of the turbulent boundary layer.

Assuming that $\ln\frac{\delta}{\delta_L} / \ln\frac{\delta_0}{\delta_{L0}} \approx 1$. approximately, we obtain from (15) and (16)

$$N(1-\varphi_{L0}) = 2\sqrt{c_f^* + N} - \sqrt{c_f^* + N\varphi_{L0}\varphi_L^*}. \quad (25)$$

Simultaneous solution of this last equation and (13) gives c_f^* and φ_L^* as functions of N and φ_{L0}. In engineering calculations the influence of blowing may be evaluated according to (25) and (13) approximately, and more accurately from (23) and (13).

NOTATION

x, y) longitudinal and transverse coordinates; v_x, v_y) longitudinal and transverse velocity components; u) free steam velocity; v_ω) blowing velocity; ρ) density; k = 0.39 and k_1 = 4.30) empirical turbulence constants; μ) viscosity; τ_ω) friction stress at the wall; Θ) momentum thickness; δ) thickness of the boundary layer; v_*) dynamic velocity. Subscripts relate to the parameters: ω) on the plate; L) at the edge of the laminar sublayer; 0) in the absence of blowing.

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Zhdanov State University, Leningrad